

# Spatiotemporal two-dimensional solitons in the complex Ginzburg-Landau equation

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## Abstract

We introduce spatiotemporal solitons of the two-dimensional complex Ginzburg-Landau equation (2D CCQGLE) with cubic and quintic nonlinearities in which asymmetry between space-time variables is included. The 2D CCQGLE is solved by a powerful Fourier spectral method, i.e., a Fourier spatial discretization and an explicit scheme for time differencing. Varying the system's parameters, and using different initial conditions, numerical simulations reveal 2D solitons in the form of stationary, pulsating and exploding solitons which possess very distinctive properties. For certain regions of parameters, we have also found stable coherent structures in the form of spinning (vortex) solitons which exist as a result of a competition between focusing nonlinearities and spreading while propagating through medium.

*Keywords:* Ginzburg-Landau, nonlinear waves, vortex solitons, spinning solitons, pulsating solitons, erupting solitons, dissipative solitons

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## 1. Introduction

The complex cubic-quintic Ginzburg-Landau equation (CCQGLE) is the canonical equation governing the weakly nonlinear behavior of dissipative systems in a wide variety of disciplines [7]. In fluid mechanics, it is also often referred to as the Newell–Whitehead equation after the authors who derived it in the context of Bénard convection [7, 8]. As such, it is also one of the most widely studied nonlinear equations. Many basic properties of the equation and its solutions are reviewed in [5, 6], together with applications to a vast variety of phenomena including nonlinear waves, superconductivity,

Bose–Einstein condensation, liquid crystals and string theory. In particular, in nonlinear optics it describes the pulse generation and signal transmission through an optical fiber.

An important element in the long time dynamics of pattern forming systems is a class of solutions which we call “coherent structures” or solitons. These structures could be a profile of light intensity, temperature, magnetic field, etc. A dissipative soliton is localized and exists for an extended period of time. Its parts are experiencing gain/loss of energy or mass with the medium.

In contrast to the regular solitary waves investigated in numerous integrable and non-integrable systems over the last three decades, these dissipative solitons are not stationary in time [2]. Rather, they are spatially confined pulse-type structures whose envelopes exhibit complicated temporal dynamics. Another relevant feature of dissipative systems is that they include energy exchange with external sources. Thus, energy and matter can flow into the system through its boundaries. As long as the parameters in the system stay constant, the structure that could evolve by changing shape exists indefinitely in time. The structure disappears when the source is switched off, or if the parameters are moved outside of the range of existence of the soliton solutions. Since these dissipative systems include energy exchange with external sources, they are no longer Hamiltonian, and the soliton solutions in these systems are also qualitatively different from those in Hamiltonian systems. In Hamiltonian systems, soliton solutions appear as a result of balance between diffraction (dispersion) and nonlinearity. Diffraction spreads a beam while nonlinearity will focus it and make it narrower. The balance between the two results in stationary solitary wave solutions, which usually form a one parameter family. In dissipative systems with gain and loss, in order to have stationary solutions, the gain and loss must be also balanced. This additional balance results in solutions which are fixed. Then the shape, amplitude and the width are all completely fixed by the parameters of the dissipative equation. However, the solitons, when they exist, can again be considered as “modes” of dissipative systems just as for non-dissipative ones [2].

When a system is integrable, its solutions can be constructed using the inverse scattering technique [1]. Further generalization to non-integrable systems shows that traveling wave solutions can be found and analyzed using some basic approaches from nonlinear dynamics. One of the powerful techniques that can be applied to these systems is that which employs Hamil-

tonian equations. For dissipative systems, there are no conserved quantities that can be used to solve particular problems.

Recent perturbative treatments based on expressions about the nonlinear Schrödinger equations are generalized to perturbations of the cubic-quintic and derivative Schrödinger equations. The cubic Ginzburg-Landau admits a selected range of exact soliton solutions [9]. Even the known special solutions have exotic dynamic responses to external conditions, so that sensitivity analysis of the parameters is paramount. These exist when certain relations between parameters are satisfied. However, this certainly does not imply that the equations are integrable. In reality, general dissipative nonlinear PDEs cannot be reduced to linear equations in any known way, therefore an insight to the type of solutions that the equation has may be based on numerical simulations. To investigate numerically the two dimensional dissipative soliton solutions of the CCQGLE, and analyze their qualitative behavior, we used a powerful Fourier spectral method.

## 2. 2D CCQGLE

The 2D CCQGLE is solved by spectral methods, i.e., a Fourier spatial discretization and an explicit scheme for time differencing. In addition, it is also crucial to analyze the dependence of both their shape and stability on the various parameters of the CCQGLE, viz. the nonlinearity, dispersion, linear and nonlinear gain, loss and spectral filtering parameters of these new classes of dissipative solitons, since the solutions experience interesting bifurcation sequences as the parameters of the CCQGLE are varied. A rich variety of evolution behaviors, which include pulsating, exploding and spinning dissipative soliton dynamics is revealed.

In dimensionless form, the 2D CCQGLE with the corresponding cubic-quintic terms is a modified NLS and it takes the form [4, 2]

$$\partial_t A = \epsilon A + (b_1 + ic_1)\nabla_{\perp}^2 A - (b_3 - ic_3)|A|^2 A - (b_5 - ic_5)|A|^4 A, \quad (2.1)$$

where  $\nabla_{\perp}^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right)$  is the transverse Laplacian for radially symmetric beams. The present study will confine itself to the spatially infinite system in two dimensions and will focus primarily on a spatiotemporal behavior of dissipative solitons without imposing radial symmetry on the beams, and hence the Laplacian  $\nabla_{\perp}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$  depends on both variables  $x$  and  $y$ .

The interpretation of the system's parameters in (2.1) depends on the

particular field of work. In optics,  $A(r; t)$  is the normalized envelope of the field,  $r$  is the radial coordinate,  $t$  is the propagation distance or the cavity number. The system's parameters of (2.1) are:  $\epsilon$  linear loss/gain,  $b_1$ - angular spectral filtering,  $c_1 = 0.5$ - second-order diffraction coefficient,  $b_3$ - nonlinear gain/loss,  $c_3 = 1$ - nonlinear dispersion,  $b_5$ - saturation of the nonlinear gain/loss, and  $c_5$ - saturation of the nonlinear refractive index.

### 3. Fourier Spectral Method

The Fourier transform of a function  $u(x, y)$  is defined by

$$\mathcal{F}(u)(k_x, k_y) = \widehat{u}(k_x, k_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(k_x x + k_y y)} u(x, y) dx dy, \quad (3.1)$$

with the corresponding inverse

$$\mathcal{F}^{-1}(\widehat{u})(x, y) = u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(k_x x + k_y y)} \widehat{u}(k_x, k_y) dk_x dk_y. \quad (3.2)$$

The function  $\widehat{u}(k_x, k_y)$  can be interpreted as the amplitude density of  $u$  for wavenumbers  $k_x, k_y$ .

First, we write the Fourier transform of equation (2.1) as

$$\widehat{A}_t = [\epsilon - (b_1 + ic_1)(k_x^2 + k_y^2)] \widehat{A} - (b_3 - ic_3) \widehat{|A|^2 A} - (b_5 - ic_5) \widehat{|A|^4 A}, \quad (3.3)$$

and rearranging the terms in (3.3), we consider the following ordinary differential equation in the Fourier space

$$\widehat{A}_t = \alpha(k_x, k_y) \widehat{A} + \beta \widehat{|A|^2 A} + \gamma \widehat{|A|^4 A}, \quad (3.4)$$

with  $\alpha(k_x, k_y) = \epsilon - (b_1 + ic_1)(k_x^2 + k_y^2)$ ,  $\beta = -(b_3 - ic_3)$  and  $\gamma = -(b_5 - ic_5)$ . In the Fourier space, (3.4) contains the linear part  $\alpha(k_x, k_y) \widehat{A}$ , and the non-linear part  $\beta \widehat{|A|^2 A} + \gamma \widehat{|A|^4 A}$ .

We solve the initial value problem for the above ODE numerically using an explicit scheme i.e., 4<sup>th</sup> order Adams-Bashforth for the non-linear part and the exact solution for the linear part, which is given by

$$\widehat{A}(t) = \widehat{A(x, y; 0)} e^{\alpha(k_x, k_y)t}. \quad (3.5)$$



### 3.1. Spatial discretization (Discrete Fourier Transform)

We discretize the spatial domain  $[-L/2, L/2] \times [-L/2, L/2]$  into  $n \times n$  uniformly spaced grid points  $X_{ij} = (x_i, y_j)$  with  $\Delta x = \Delta y = L/n$  and  $n$  even. Given  $A(X_{ij}) = A_{ij}$ ,  $i, j = 1, 2, \dots, n$ , we define the 2D Discrete Fourier transform (2DFT) of  $A$  as

$$\hat{A}_{k_x k_y} = \Delta x \Delta y \sum_{i=1}^n \sum_{j=1}^n e^{-i(k_x x_i + k_y y_j)} A_{ij}, \quad k_x, k_y = -\frac{n}{2} + 1, \dots, \frac{n}{2} \quad (3.6)$$

and its inverse 2DFT as

$$A_{ij} = \frac{1}{(2\pi)^2} \sum_{k_x=-n/2+1}^{n/2} \sum_{k_y=-n/2+1}^{n/2} e^{i(k_x x_i + k_y y_j)} \hat{A}_{k_x k_y}, \quad i, j = 1, 2, \dots, n. \quad (3.7)$$

In (3.6)-(3.7) the wavenumbers  $k_x$  and  $k_y$ , and the spatial indexes  $i$  and  $j$ , take only integer values.

### 3.2. Temporal discretization (4<sup>th</sup> order Adams-Bashforth)

Given  $t_{\max}$  we discretize the time domain  $[0, t_{\max}]$  with equal time steps of width  $\Delta t$  as  $t_n = n\Delta t$ ,  $n = 0, 1, 2, \dots$ , and define  $\hat{A}^n = \hat{A}(t_n)$ . Initializing  $\hat{A}^n = \hat{A}(t_n)$ , we compute the nonlinear terms  $\mathcal{N}_3 = \mathcal{F} \left( \left| \mathcal{F}^{-1}(\hat{A}^n) \right|^2 \mathcal{F}^{-1}(\hat{A}^n) \right)$ , and  $\mathcal{N}_5 = \mathcal{F} \left( \left| \mathcal{F}^{-1}(\hat{A}^n) \right|^4 \mathcal{F}^{-1}(\hat{A}^n) \right)$  and advance the ODE (3.4) in time with time step  $\Delta t$  using the 4-step Adams-Bashforth method.

The discretized ODE has the following form

$$\hat{A}^{n+1} = \hat{A}^n e^{\alpha(k_x, k_y)t} + \frac{\Delta t}{24} \left[ 55f(\hat{A}^n) - 59f(\hat{A}^{n-1}) + 37f(\hat{A}^{n-2}) - 9f(\hat{A}^{n-3}) \right], \quad (3.8)$$

with exact solution of the linear part  $\hat{A}^{n+1} = \hat{A}^n e^{\alpha(k_x, k_y)t}$ , and the nonlinear part  $f$  defined by  $f(\hat{A}) = \beta \mathcal{N}_1 + \gamma \mathcal{N}_2$ .

### 3.3. Numerical implementation

The language used is Fortran90, with gfortran compiler. All the parameters needed by the program are written in a datafile so that they can be easily modified, without systematic compilation. Among them, most important are the parameters of the 2D CCQGLE, which are shown in Table 1. For

2D solitons	$\epsilon$	$b_1$	$c_1$	$b_3$	$c_3$	$b_5$	$c_5$
stationary	-0.045	0.04	0.5	-0.21	1	0.03	-0.08
ring	-0.1	0.1	0.5	-0.8	1	0.04	-0.02
pulsating	-0.045	0.04	0.5	0.37	1	0.05	-0.08

Table 1: Sets of parameters for 2D solitons

the ring solitons from §4.4 we have used the parameters of [3], while for the stationary and pulsating, the parameters are from [12].

In this paper we are recalculating the results published previously in [3, 12], and using this scheme, we hope to find other regions of parameters where stable 2D solitons exist, as well as solitons with arbitrary vorticity ( $m$ ).

An important subroutine is the Fast Fourier Transform (FFT) [10, 15]. This subroutine uses an algorithm that reduces computational time from  $\mathcal{O}(n^2)$  to  $\mathcal{O}(n \log n)$  floating point operations. Moreover, it only works for periodic functions hence, caution must be made when we select the size of the domain for different cases of the soliton solutions. This must be chosen in such way that the solitons do not touch the boundaries, otherwise if the non-zero part of the solutions propagates through some boundary, it will spread throughout the domain from the opposite side.

The Fortran program ran on the Zeus cluster at Embry-Riddle Aeronautical University. This parallel computer of 256 nodes is rated to over 1 Tera-flop. Jobs were submitted thanks to Platform Lava installed on the cluster. This software manages to distribute all the jobs to different nodes, considering priorities. Once the output file are written, we used Tecplot 360 for visualization.

The duration of computations depends on the equation parameters and mesh size. The most appropriate size to use was 512x512, but other types of meshes were considered. Very fast calculations took almost two hours, while longer ones took several days.

## 4. Numerical Simulations

### 4.1. Initial conditions

Localized structures with vorticity are obtained when initial conditions have radial symmetry, and hence we assume that the initial pulse can be

reasonably well approximated by a bell-shaped, i.e.,

- i. Gaussian shape,

$$A(x, y; 0) = A_0 e^{-r^2} \quad (4.1)$$

- ii. Ring shape with rotating phase,

$$A(x, y; 0) = A_0 r^m e^{-r^2} e^{im\theta} \quad (4.2)$$

where  $m$  is the degree of vorticity,  $A_0$  is a real amplitude that should generate sufficient power to place the initial condition into a basin of attraction of a 2D soliton, and  $\theta = \tan^{-1} \left( \frac{\sigma_y y}{\sigma_x x} \right)$  is the phase. The widths of each of the initial conditions could be either circular (radially symmetric) or elliptic, and are controlled by the inverse widths parameters  $\sigma_x$  and  $\sigma_y$ , with  $r = \sqrt{(\sigma_x x)^2 + (\sigma_y y)^2}$ , see Fig. 1. In this paper we have used the vorticity  $m = 1$ , but family of solitons of higher vorticity have been found [14].

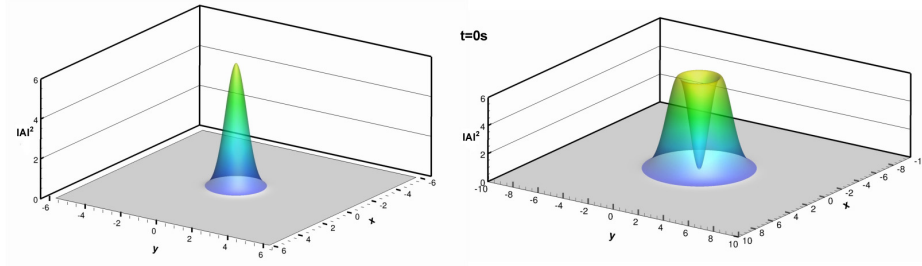


Figure 1: Initial shapes of the field. Left: Gaussian shape. Right: Ring shape with vorticity  $m = 1$ .

#### 4.2. Parameters

The existence of stable solutions is strongly linked to the parameters of the equation. One of the most important issue is to find the continuous range of existence of solitons of the same type in parameters space. Moreover, several solutions can coexist for the same set of parameters. These solitons are not necessarily stable. Hence, stability is controlled by the parameters of the equation and by the choice of the initial conditions.

The main parameter of the solution that we monitor in our simulation is the energy [12],  $Q(t)$  where

$$Q(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |A(x, y; t)|^2 dx dy = \sum_{i=1}^n \sum_{j=1}^n |A_{ij}|^2 \Delta x \Delta y \quad (4.3)$$

and is a conserved quantity (a dynamical invariant of (2.1)). For a localized solution  $Q$  is finite and changes smoothly while the solution stays within the region of existence of the soliton. When  $Q$  changes abruptly then there is a bifurcation and the solution jumps from a branch of solitons that become unstable to another branch of stable solitons. Thus, monitoring  $Q$  allows us to find bifurcations in an easy way. Observing a finite  $Q$  also reveals the stability of a solution. As soon as the solution becomes unstable,  $Q$  diverges. In the case that the resulting solution is pulsating, if the solution stays localized, the energy will evolve periodically and will remain finite. When the optical field spreads out, the energy tends to infinity, and when the solution dissipates the energy evolves to zero [12].

#### 4.3. Stationary solitons

The initial shape is a circular 2D Gaussian with  $A_0 = 2.5$ , and  $\sigma_x = \sigma_y = 1$ . Steady solitons are the simplest solutions we can find. After a transition period of stabilization, the shape of the solitary waves remains the same. The energy becomes constant after  $z \approx 40s$ . The soliton profile is always a Gaussian, and each profile has a plain bell-shape. Finally, energy converges to  $Q(t) \approx 18.6$ , and stabilizes forever if the system's parameters are not changed.

#### 4.4. Ring vortex (spinning) solitons

We choose a circular ring type structure with vorticity  $m = 1$ ,  $A_0 = 2.5$ , and  $\sigma_x = \sigma_y = 1$ , and parameters from line 2 of Table 1. By keeping all the parameters fixed, we can find different types of ring vortex solitons, by varying the initial conditions. This also can be achieved by varying slowly one of the equation parameters while the initial condition stays fixed. First, we found a radially symmetric ring vortex soliton shown in Fig. 3. Even though the structure oscillates before converging to a stable value of the energy as it is with the stable soliton, now since we introduced a phase, the soliton is spinning. The rotation has a fixed angular velocity and can occur

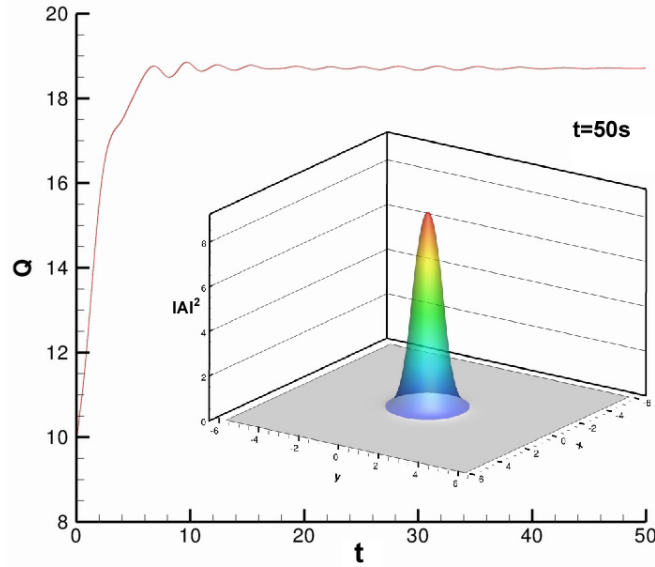


Figure 2: Stationary soliton. Energy is concentrated in the center of the domain, in a bell-shaped structure.

in either direction clockwise or counterclockwise, which is determined by the initial condition.

Next, we introduced for the same set of parameters, a different initial condition, viz. an elliptic shape which breaks the symmetry,  $\sigma_x = 0.15$ ,  $\sigma_y = 0.85$ . This new initial condition leads to another kind of soliton with an elongated shape and two peaks diametrically opposed at the top of the structure.

The results for both radially symmetric and elliptic solitons are shown in Figs. 3 and 4.

Ordinary bell-shaped solitons with zero-vorticity also exist for the same parameters, but with lower energy. If a ring vortex soliton loses its stability, it will be transformed into several bell-shaped solitons via multiple bifurcations. That is what we have found, by chance, while looking for the pulsating ring vortex soliton, see Fig. 5. This class appears to be unstable, may lead to chaos, and needs further investigation.

#### 4.5. Pulsating solitons

The following results contain two interesting features: pulsations and transitions between stable and unstable states. The initial shape is also

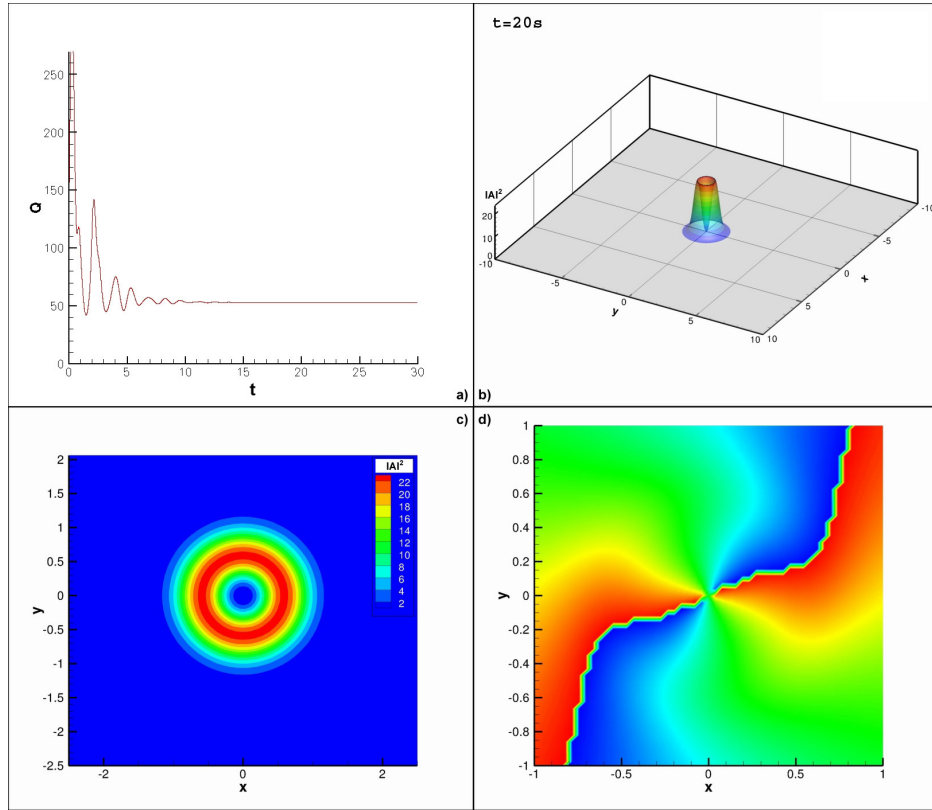


Figure 3: Circular ring vortex soliton with radially symmetry. Top left: Energy. Top right: Ring vortex soliton at  $t = 20s$ . Bottom left: Contour plot of  $|A|^2$ . Bottom right: Phase plot that indicates rotation at  $t = 20s$ .

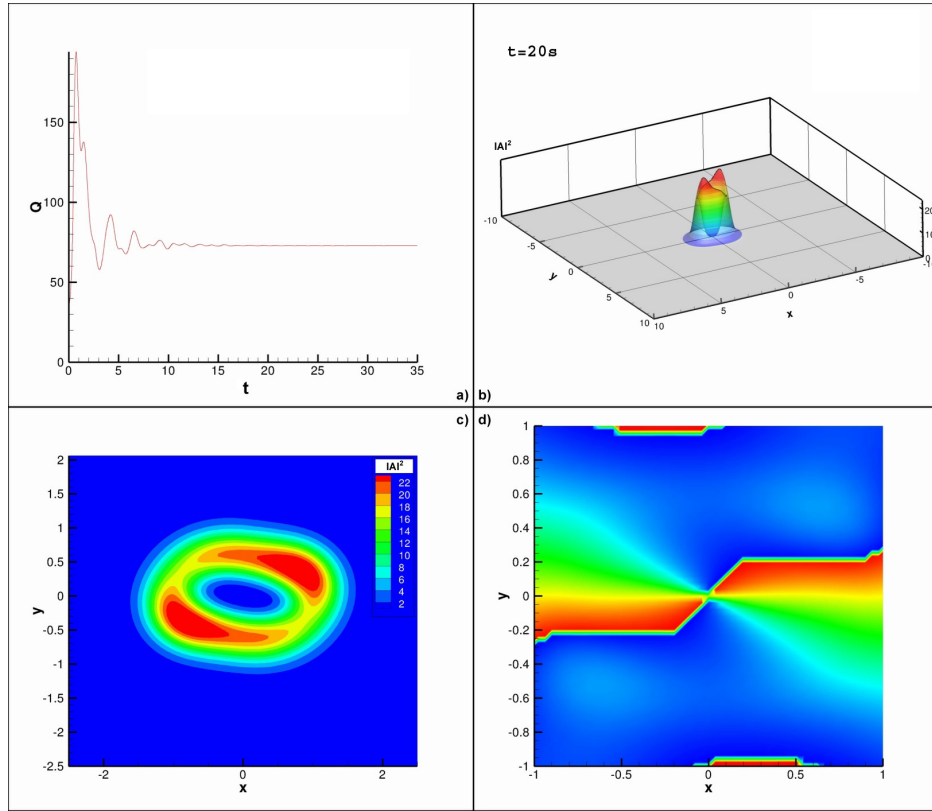


Figure 4: Elliptic ring vortex soliton (no radial symmetry). Two peaks appear at the top of the structure, while the phase keeps spinning.

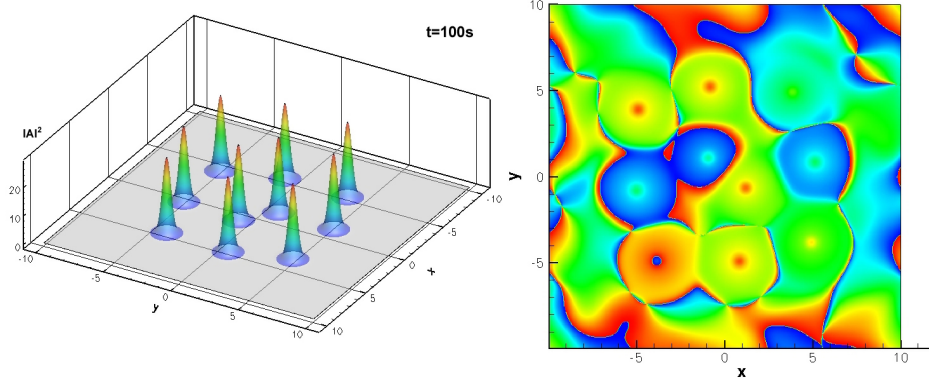


Figure 5: Unstable vortex circular ring. Left: 10 bell-shaped solitons appeared in results to the destruction of the initial ring soliton. Right: the phases of the solitons are not spinning. The parameters of the initial shape are:  $A_0 = 3.0$ ,  $\sigma_x = \sigma_y = 0.15$ .

Gaussian with  $A_0 = 5$ , but slightly asymmetric,  $\sigma_x = 0.8333$  and  $\sigma_y = 0.9091$ . Here we have used the parameters from line 3 of Table 1. For the first  $200s$ , the evolution of the solution looks similar to the stationary soliton, see Fig.6. While the soliton is stabilized at  $t = 430s$ , a perturbation or a bifurcation appears to make the structure evolve toward a new higher state: the energy  $Q$  of the soliton is now pulsating. The structure still has radially symmetric profile when  $Q$  takes its maximum, but elongates alternatively in different directions when  $Q$  takes its minimum value, see Fig.7.

Indeed, this state is not stable for this set of parameters and evolves into another stable soliton which keeps the elongated shape, as if it was a double bell-shaped soliton. The energy still pulsates but with a very little amplitude and the soliton rotates around its center. Different types of pulsating solitons, with varying energies and shapes, that rotate or not, have been identified, but there is no doubt that there are other types exist too, especially when two or more solitons coexist in the same plane at the same time.

#### 4.6. Exploding solitons

Another class of 2D solitons is the exploding or erupting solitons. The evolution starts from a localized initial condition of Gaussian shape with parameters  $A_0 = 3.0$ ,  $\sigma_x = \sigma_y = 0.3$ . After a while its slopes become covered with small scale instabilities which seem to move downwards. The soliton explodes intermittently, thus resulting in significant bursts of power above the



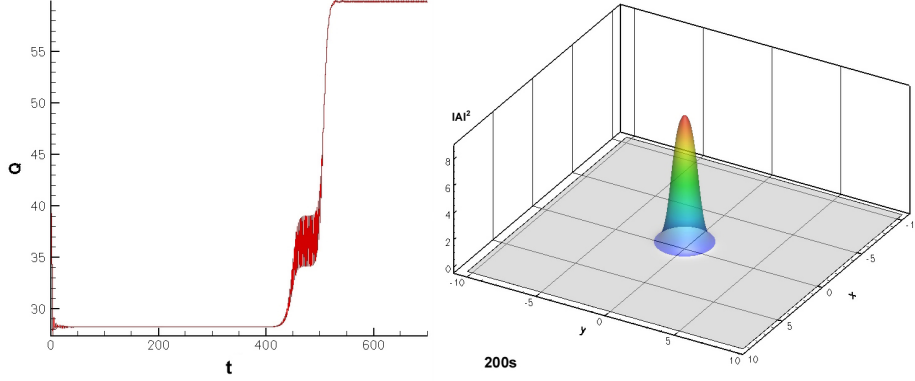


Figure 6: Pulsating soliton with changing state. The energy, on the left shows these transitions. Right: bell-shaped soliton  $t = 200s$ .

average, recovering the initial radially cylindrical shape after each explosion [4]. Using the same parameters as in the case of the 1D case of papers [13, 11], i.e.,  $\epsilon = -0.1, b_1 = 0.125, c_1 = 0.5, b_3 = -1, c_3 = 1, b_5 = 0.1, c_5 = -0.6$ , the energy  $Q$  for an exploding soliton existing in the 2D case is shown in Fig. 8. As it was shown for the 1D exploding solitons, this unusual dynamics appears as a result of an instability. As we move further from this boundary, the explosions become more violent, and, as a result more than one beam can be generated in some cases. This gives rise to a very complicated dynamics, ending up with the whole numerical grid filled with the solution.

At first, the shape seems to be stable with constant energy and smooth bell-shaped cross section ( $t \approx 28s$ ). Often, circular waves go from the center to the exterior and vanish as if energy were dissipated in the boundaries of the solitons ( $t \approx 32s$ ). Suddenly, and periodically, the soliton “explodes” ( $t \approx 35s$ ): its shape grows, so do the energy, until it collapses and goes back to the original shape ( $t \approx 36s$ ). This completely chaotic, but well-localized structure restores to a perfect solitons shape after the main burst. The periodic evolution of the energy, with high bursts of energy can be seen in Fig. 8. This process never repeats itself in successive periods, however, it always returns to the same shape. Hence, the essential features of the explosions, both observed experimentally in 1D by [11, 3] are: explosions occur intermittently, the explosions have similar features but are not identical, and explosions happen spontaneously being triggered by perturbations. These

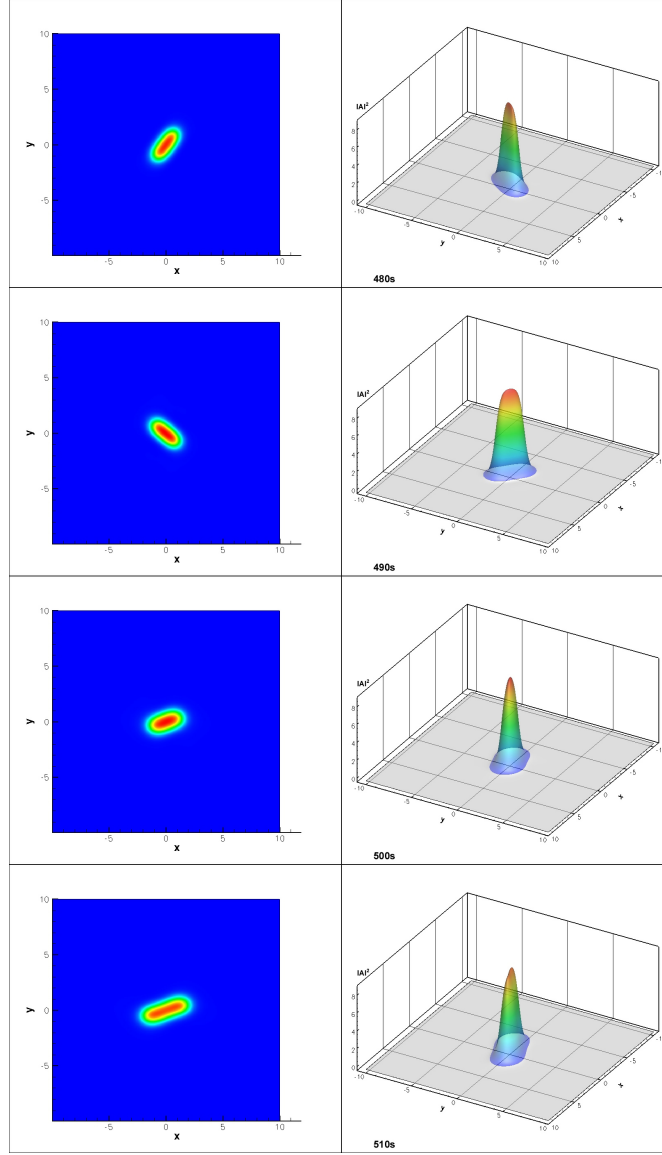


Figure 7: During the pulsating state  $t \in [480s, 520s]$  the soliton elongates alternatively in different directions.

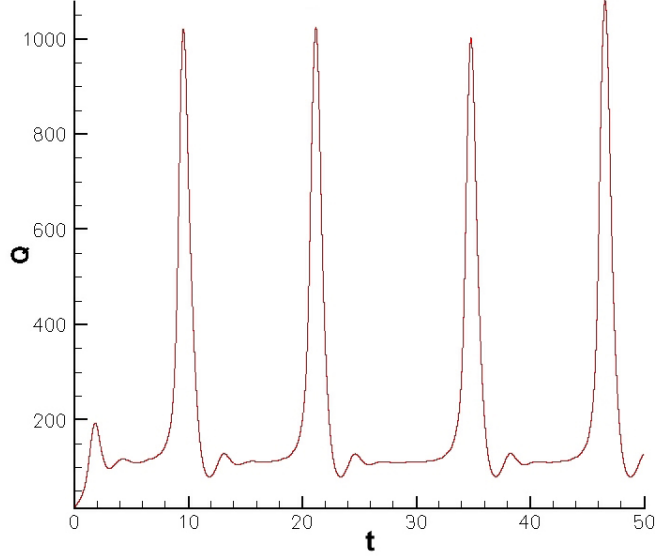


Figure 8: Exploding soliton: energy is periodic with high bursts almost every  $12s$ .

type of solitons need more numerical studies to be understood in their full complexity, and in the future we hope to find other parameters of existence for the 2D exploding soliton.

## 5. Conclusion

In conclusion, based on the numerical simulations, we have presented evidence for the existence of stable stationary 2D solitons in a dissipative medium, described by the 2D CCQGLE, in which asymmetry between space-time variables is included. For the domains of the parameters that we explored, we have also found stable coherent structures in the form of spinning solitons. These exist for a certain region of parameters, as a result of a competition between focusing nonlinearities and spreading while propagating through medium. Beyond the regions of stability, the solitons may lose the radial symmetry but remain stable. Varying the parameters, and using Gaussian and ring initial conditions, the simulations revealed more distinct 2D solitons in the form of pulsating, and for the first time two-dimensional exploding solitons with very interesting properties.

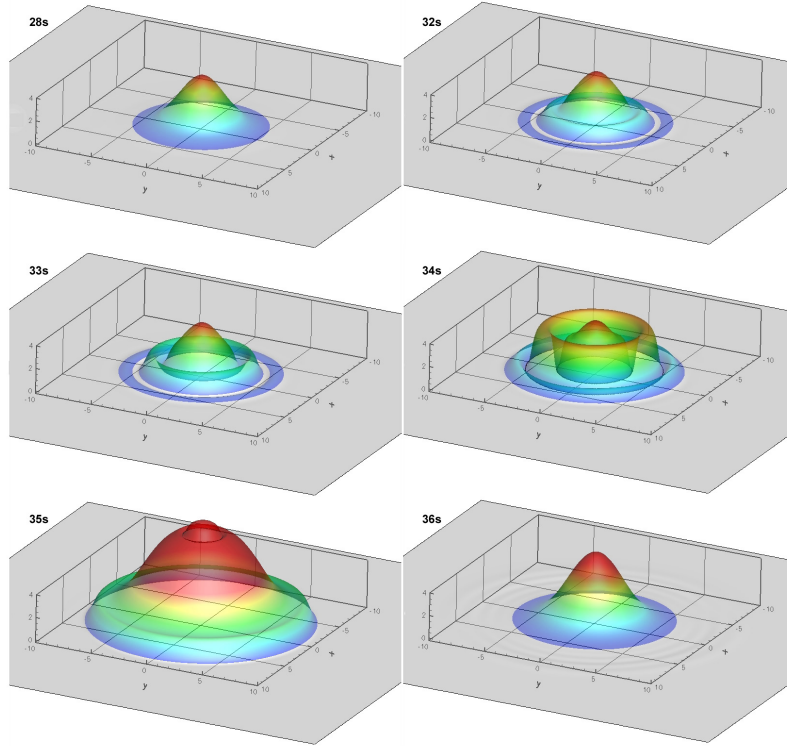


Figure 9: Exploding soliton at  $t = 28s$ , the shape is smooth. Then, circular waves appear and grow. Finally, soliton explodes around  $t = 34s$  and collapses after which it restores back to its initial shape.

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